

On the Relativistic Invariance of Maxwell's Equation

Oleg D. Jefimenko

Physics Department, West Virginia University, P.O. Box 6315, Morgantown, WV 26506-6315, USA

Reprint requests to Prof. O. D. J.; Fax: 304293-5732.

Z. Naturforsch. **54a**, 637–644 (1999); received September 25, 1999

It is common knowledge that Maxwell's electromagnetic equations are invariant under relativistic transformations. However the relativistic invariance of Maxwell's equations has certain heretofore overlooked peculiarities. These peculiarities point out to the need of reexamining the physical significance of some basic electromagnetic formulas and equations.

Key words: Maxwell's Equation; Special Relativity; Lorentz Contraction.

1. Introduction

Maxwell's electromagnetic equations are the four differential equations

$$\nabla \cdot \mathbf{D} = \rho, \quad (1)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (2)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (3)$$

and

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \quad (4)$$

where \mathbf{E} is the electric field vector, \mathbf{D} the electric displacement vector, \mathbf{H} the magnetic field vector, \mathbf{B} the magnetic flux density vector, \mathbf{J} the electric current density vector, and ρ the electric charge density. For fields in a vacuum (the only fields with which we shall be concerned in this paper), Maxwell's equations are supplemented by the two "constitutive equations"

$$\mathbf{D} = \epsilon_0 \mathbf{E} \quad (5)$$

and

$$\mathbf{B} = \mu_0 \mathbf{H}, \quad (6)$$

where ϵ_0 is the permittivity of space and μ_0 is the permeability of space.

As is known, in the special relativity theory one distinguishes between the "proper" charge density ρ_0 , measured when the charge under consideration is at rest relative to the observer, and the "nonproper", or "relativistic", or "Lorentz-contracted" charge density defined as

$$\rho = \rho_0 (1 - u^2/c^2)^{-1/2}, \quad (7)$$

where u is the velocity of the charge under consideration relative to the reference frame from which ρ is observed.

Maxwell's equations were developed many years before the advent of the theory of special relativity, and ρ in the prerelativistic equation (1) represented the electrostatic charge density even if the charge under consideration was moving [1]. Since the electrostatic charge density is the same as the proper charge density ρ_0 of the theory of special relativity, we can write (1) as

$$\nabla \cdot \mathbf{D} = \rho_0. \quad (8)$$

Naturally, it is possible to write (1) so that ρ in it becomes the relativistic charge density. To do so one only needs to replace ρ in (1) by the right side of (7), which gives

$$\nabla \cdot \mathbf{D} = \rho_0 (1 - u^2/c^2)^{-1/2}. \quad (9)$$

It is common knowledge that Maxwell's equations are invariant under relativistic transformations. But is (1) invariant when written as (8) or is it invariant when written as (9)? In this connection it is important to note that Einstein (as well as Lorentz, Larmor, and Poincaré, who preceded Einstein in the development and use of relativistic transformations and in the demonstration of the invariance of Maxwell's equations under these transformations) used prerelativistic Maxwell's equations, in which the charge density was meant to be the electrostatic charge density. Therefore, in terms of the notation defined above, the pioneers of relativity only showed that (8) was invariant under relativistic transformations, but they did not consider (9) or any equation equivalent to (9) at all. Moreover, even now, the authors of the very few textbooks on relativity and electromagnetic theory in which the invariance of Maxwell's equations is actually shown use Maxwell's equations in their original form

0932-0784 / 99 / 1000-0637 \$ 06.00 © Verlag der Zeitschrift für Naturforschung, Tübingen · www.znaturforsch.com



Dieses Werk wurde im Jahr 2013 vom Verlag Zeitschrift für Naturforschung in Zusammenarbeit mit der Max-Planck-Gesellschaft zur Förderung der Wissenschaften e.V. digitalisiert und unter folgender Lizenz veröffentlicht: Creative Commons Namensnennung-Keine Bearbeitung 3.0 Deutschland Lizenz.

Zum 01.01.2015 ist eine Anpassung der Lizenzbedingungen (Entfall der Creative Commons Lizenzbedingung „Keine Bearbeitung“) beabsichtigt, um eine Nachnutzung auch im Rahmen zukünftiger wissenschaftlicher Nutzungsformen zu ermöglichen.

This work has been digitalized and published in 2013 by Verlag Zeitschrift für Naturforschung in cooperation with the Max Planck Society for the Advancement of Science under a Creative Commons Attribution-NoDerivs 3.0 Germany License.

On 01.01.2015 it is planned to change the License Conditions (the removal of the Creative Commons License condition "no derivative works"). This is to allow reuse in the area of future scientific usage.

[2] [that is, they actually use (8) rather than (9)] and, what is most important, use as the transformation equation for ϱ the equation

$$\varrho = \gamma [\varrho' + (v/c^2) J'_x] \quad (10)$$

rather than the equation

$$\varrho_0 (1 - u^2/c^2)^{-1/2} = \gamma [\varrho'_0 (1 - u'^2/c^2)^{-1/2} + (v/c^2) J'_x], \quad (11)$$

where the Lorentz contraction of moving charges is explicitly taken into account [the unprimed quantities in these equations refer to the stationary reference frame Σ , the primed quantities refer to the moving reference frame Σ' ; u is the velocity of the charge measured in Σ , u' is the velocity of the charge measured in Σ' , v is the velocity of Σ' relative to $\Sigma = (1 - v^2/c^2)^{-1/2}$].

The purpose of this paper is to determine which particular "Maxwell's equations" are invariant under relativistic transformations and to investigate whether by expressing (1) in the form of (8) or (9) we may discover some heretofore neglected consequences of the relativistic invariance of Maxwell's equations. To make clear the distinction between the prerelativistic Maxwell's equations and Maxwell's equations incorporating Lorentz-contracted charge densities, we shall use in the discussion that follows (8) or (9) in lieu of (1). We shall then refer to (8), (2), (3), and (4), when used together, as the "original Maxwell's equations;" similarly, we shall refer to (9), (2), (3), and (4), when used together, as the "relativistic Maxwell's equations". Furthermore, for reasons that will become clear later, we shall refer to ϱ_0 as the "electrostatic charge density".

2. Invariance of the Original Maxwell's Equations under Relativistic Transformations

Let us first show that the original Maxwell's equations (with $\varrho = \varrho_0$ representing the electrostatic charge density) remain invariant under relativistic transformations. For this purpose we shall use (8), (2), (3), (4) and transformation equations listed in the Appendix (the latter equations are identified by the prefix "A"). Observe that two types of transformation equations for charge and current densities are given in the Appendix. Equations (A11)–(A13) are the conventional equations except for the subscript "0", which indicates that we are using them specifically for transforming electrostatic charge densities (the only charge densities known in the prerelativistic electromagnetism). Equations (A22)–(A24) are equations written explicitly for the Lorentz-contracted charge densities.

Transformation of $\nabla \cdot \mathbf{D} = \varrho_0$. Remembering that, by (5), $\mathbf{D} = \varepsilon_0 \mathbf{E}$ and writing Maxwell's equation (8) in terms of Cartesian components, we have

$$\varepsilon_0 \frac{\partial E_x}{\partial x} + \varepsilon_0 \frac{\partial E_y}{\partial y} + \varepsilon_0 \frac{\partial E_z}{\partial z} = \varrho_0. \quad (12)$$

Using (A17), (A1), (A18), (A3), (A19), (A5), and (A12), we can write (12) as

$$\begin{aligned} & \varepsilon_0 \frac{\partial E'_x}{\gamma \partial x'} - \varepsilon_0 \frac{v}{c^2} \frac{\partial E_x}{\partial t} + \varepsilon_0 \frac{\partial E'_y}{\gamma \partial y'} + \varepsilon_0 v \frac{\partial B_z}{\partial y} \\ & + \varepsilon_0 \frac{\partial E'_z}{\gamma \partial z'} - \varepsilon_0 v \frac{\partial B_y}{\partial z} = \frac{1}{\gamma} \varrho'_0 + \frac{v}{c^2} J_x. \end{aligned} \quad (13)$$

Rearranging, we have

$$\begin{aligned} & \frac{1}{\gamma} \left(\frac{\partial \varepsilon_0 E'_x}{\partial x'} + \frac{\partial \varepsilon_0 E'_y}{\partial y'} + \frac{\partial \varepsilon_0 E'_z}{\partial z'} \right) = \frac{1}{\gamma} \varrho'_0 \\ & - \varepsilon_0 v \left(\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} \right) + \frac{v}{c^2} \left(J_x + \frac{\partial \varepsilon_0 E_x}{\partial t} \right). \end{aligned} \quad (14)$$

However, since, by (6), $\mathbf{B} = \mu_0 \mathbf{H}$, since $\varepsilon_0 \mu_0 = 1/c^2$, and since $\varepsilon_0 \mathbf{E} = \mathbf{D}$, the last two terms in (14) are simply the x component of the expression

$$\frac{v}{c^2} \left(-\nabla \times \mathbf{H} + \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right), \quad (15)$$

which, by Maxwell's equation (4), is zero. Hence, dropping the last two terms in (14), cancelling γ , replacing $\varepsilon_0 \mathbf{E}'$ by \mathbf{D}' , and restoring the vector notation, we obtain

$$\nabla' \cdot \mathbf{D}' = \varrho'_0. \quad (16)$$

Thus the original Maxwell's equation (8) is invariant (retains its form) under relativistic transformations with (A12) used for the transformation of the charge density.

Transformation of $\nabla \cdot \mathbf{B} = 0$. Writing Maxwell's equation (2) in terms of Cartesian components, we have

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0. \quad (17)$$

Using (A17), (A6), (A18), (A8), (A19), and (A10), we can write (17) as

$$\begin{aligned} & \frac{\partial B'_x}{\gamma \partial x'} - \frac{v}{c^2} \frac{\partial B_x}{\partial t} + \frac{\partial B'_y}{\gamma \partial y'} - \frac{v}{c^2} \frac{\partial E_z}{\partial y} \\ & + \frac{\partial B'_z}{\gamma \partial z'} + \frac{v}{c^2} \frac{\partial E_y}{\partial z} = 0. \end{aligned} \quad (18)$$

Multiplying by γ and rearranging, we have

$$\begin{aligned} & \frac{\partial B'_x}{\partial x'} + \frac{\partial B'_y}{\partial y'} + \frac{\partial B'_z}{\partial z'} \\ &= \gamma \frac{v}{c^2} \left[\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) + \frac{\partial B_x}{\partial t} \right]. \end{aligned} \quad (19)$$

However, the expression in the brackets is simply the x component of the expression

$$\frac{v}{c^2} \left(\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \right), \quad (20)$$

which, by Maxwell's equation (3), is zero. Replacing the right side of (19) by zero and restoring the vector notation, we obtain

$$\nabla' \cdot \mathbf{B}' = 0 \quad (21)$$

Thus Maxwell's equation (2) is invariant under relativistic transformations and, since in deriving (21) we did not use transformation equations for charge or current densities, the invariance of (2) does not depend on the transformation equations for charge or current densities.

Transformation of $\nabla \times \mathbf{E} = -\partial \mathbf{B} / \partial t$. Writing Maxwell's equation (3) in terms of Cartesian components, we have

$$\begin{aligned} & i \left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \right) + j \left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \right) \\ &+ k \left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \right) = -i \frac{\partial B_x}{\partial t} - j \frac{\partial B_y}{\partial t} - k \frac{\partial B_z}{\partial t}. \end{aligned} \quad (22)$$

Using (A18), (A19), (A1), (A2), (A4), (A6), and (A20), we can write (22) as

$$\begin{aligned} & i \left(\gamma \frac{\partial E'_z}{\partial y'} - \gamma v \frac{\partial B'_y}{\partial y'} - \gamma \frac{\partial E'_y}{\partial z'} - \gamma v \frac{\partial B'_z}{\partial z'} \right) \\ &+ j \left(\frac{\partial E'_x}{\partial z'} - \frac{\partial E_z}{\partial x} \right) + k \left(\frac{\partial E_y}{\partial x} - \frac{\partial E'_x}{\partial y'} \right) \\ &= -i \left(\gamma \frac{\partial B'_x}{\partial t'} - \gamma v \frac{\partial B'_x}{\partial x'} \right) - j \frac{\partial B_y}{\partial t} - k \frac{\partial B_z}{\partial t}. \end{aligned} \quad (23)$$

According to (21), the terms with the derivatives $\partial B'_x / \partial x'$, $\partial B'_y / \partial y'$, $\partial B'_z / \partial z'$ in (23) vanish, so that the equation simplifies to

$$\begin{aligned} & i \left(\gamma \frac{\partial E'_z}{\partial y'} - \gamma \frac{\partial E'_y}{\partial z'} \right) + j \left(\frac{\partial E'_x}{\partial z'} - \frac{\partial E_z}{\partial x} \right) \\ &+ k \left(\frac{\partial E_y}{\partial x} - \frac{\partial E'_x}{\partial y'} \right) = -i \gamma \frac{\partial B'_x}{\partial t'} - j \frac{\partial B_y}{\partial t} - k \frac{\partial B_z}{\partial t}. \end{aligned} \quad (24)$$

Using (A4), (A2), (A7), and (A9), we can write (24) as

$$\begin{aligned} & i \left(\gamma \frac{\partial E'_z}{\partial y'} - \gamma \frac{\partial E'_y}{\partial z'} \right) + j \left[\frac{\partial E'_x}{\partial z'} - \gamma \frac{\partial (E'_z - v B'_y)}{\partial x} \right] \\ &+ k \left[\gamma \frac{\partial (E_y - v B'_z)}{\partial x} - \frac{\partial E'_x}{\partial y'} \right] = -i \gamma \frac{\partial B'_x}{\partial t'} \\ &- j \gamma \frac{\partial (B'_y - v E'_z / c^2)}{\partial t} - k \gamma \frac{\partial (B'_z - v E'_y / c^2)}{\partial t} \end{aligned} \quad (25)$$

or, rearranging, as

$$\begin{aligned} & i \left(\gamma \frac{\partial E'_z}{\partial y'} - \gamma \frac{\partial E'_y}{\partial z'} \right) + j \left[\frac{\partial E'_x}{\partial z'} - \gamma \left(\frac{\partial E'_z}{\partial x} + \frac{v}{c^2} \frac{\partial E'_y}{\partial t} \right) \right] \\ &+ k \left[\gamma \left(\frac{\partial E'_y}{\partial x} + \frac{v}{c^2} \frac{\partial E'_y}{\partial t} \right) - \frac{\partial E'_x}{\partial y'} \right] = -i \gamma \frac{\partial B'_x}{\partial t'} \\ &- j \gamma \left(\frac{\partial B'_y}{\partial t} + v \frac{\partial B'_y}{\partial x} \right) - k \gamma \left(\frac{\partial B'_z}{\partial t} + v \frac{\partial B'_z}{\partial x} \right), \end{aligned} \quad (26)$$

which, by (A16) and (A21), is

$$\begin{aligned} & i \left(\gamma \frac{\partial E'_z}{\partial y'} - \gamma \frac{\partial E'_y}{\partial z'} \right) + j \left(\frac{\partial E'_x}{\partial z'} - \frac{\partial E'_z}{\partial x'} \right) \\ &+ k \left(\frac{\partial E'_y}{\partial x'} - \frac{\partial E'_x}{\partial y'} \right) = -i \gamma \frac{\partial B'_x}{\partial t'} - j \frac{\partial B'_y}{\partial t'} - k \frac{\partial B'_z}{\partial t'}. \end{aligned} \quad (27)$$

Comparing the x , y , and z components of the left side of (27) with those of the right side, we find that the components have the same form as the components of (22) (the factor γ in the x components cancels if one compares only the individual components of the left and the right side of the equation). It is interesting to note, however, that although the Cartesian components of Maxwell's equation (3) are invariant under relativistic transformations, the vector equation itself is not invariant because γ is only present in the x components of (27) but not in the y and z components [therefore γ cannot be cancelled

from (27), and (27) does not have the same form as (22) or (3)]^a.

Since in deriving (27) we did not use transformation equations for charge or current densities, the invariance of the Cartesian components of (3) does not depend on the transformation equations for charge or current densities.

Transformation of $\nabla \times \mathbf{H} = \mathbf{J} + \partial \mathbf{D} / \partial t$. Remembering that $\mathbf{D} = \epsilon_0 \mathbf{E}$ and writing Maxwell's equation (4) in terms of Cartesian components, we have

$$\begin{aligned} & i \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} \right) + j \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \right) \\ & + k \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \right) = i J_x + j J_y + k J_z \\ & + i \epsilon_0 \frac{\partial E_x}{\partial t} + j \epsilon_0 \frac{\partial E_y}{\partial t} + k \epsilon_0 \frac{\partial E_z}{\partial t}. \end{aligned} \quad (28)$$

Using (A18), (A9), (A19), (A7), (A6), (A1), (A20), and (A13)–(A15), and remembering that $\mathbf{B} = \mu_0 \mathbf{H}$, we can write (28) as

$$\begin{aligned} & i \left(\gamma \frac{\partial H'_z}{\partial y'} + \gamma v \frac{\partial E'_y}{\mu_0 c^2 \partial y'} - \gamma \frac{\partial H'_y}{\partial z'} + \gamma v \frac{\partial E'_z}{\mu_0 c^2 \partial z'} \right) \\ & + j \left(\frac{\partial H'_x}{\partial z'} - \frac{\partial H'_z}{\partial x} \right) + k \left(\frac{\partial H'_y}{\partial x} - \frac{\partial H'_x}{\partial y} \right) \\ & = i \gamma (J'_x + v \rho'_0) + j J'_y + k J'_z + i \epsilon_0 \gamma \\ & \cdot \left(\frac{\partial E'_x}{\partial t'} - v \frac{\partial E'_x}{\partial x} \right) + j \epsilon_0 \frac{\partial E_y}{\partial t} + k \epsilon_0 \frac{\partial E_z}{\partial t}. \end{aligned} \quad (29)$$

According to (16), taking into account that $1/\mu_0 c^2 = \epsilon_0$, the terms with $\partial E'_x / \partial x$, $\partial E'_y / \partial y$, $\partial E'_z / \partial z$ and ρ'_0 in (29) vanish, so that the equation simplifies to

$$\begin{aligned} & i \left(\gamma \frac{\partial H'_z}{\partial y'} - \gamma \frac{\partial H'_y}{\partial z'} \right) + j \left(\frac{\partial H'_x}{\partial z'} - \frac{\partial H'_z}{\partial x} \right) \\ & + k \left(\frac{\partial H'_y}{\partial x} - \frac{\partial H'_x}{\partial y} \right) = i \gamma J'_x + j J'_y + k J'_z \\ & + i \gamma \epsilon_0 \frac{\partial E'_x}{\partial t'} + j \epsilon_0 \frac{\partial E_y}{\partial t} + k \epsilon_0 \frac{\partial E_z}{\partial t}. \end{aligned} \quad (30)$$

^a It should also be noted that Lorentz, Larmor, Poincaré, and Einstein used Maxwell's equations in their scalar form. Therefore they only showed the invariance of the Cartesian components of Maxwell's equations but not the invariance of the vector form of Maxwell's equations.

Using (A9), (A7), (A2), and (A4), we can write (30) as

$$\begin{aligned} & i \gamma \left(\frac{\partial H'_z}{\partial y'} - \frac{\partial H'_y}{\partial z'} \right) \\ & + j \left[\frac{\partial H'_x}{\partial z'} - \gamma \frac{\partial (H'_z + v E'_y / \mu_0 c^2)}{\partial x} \right] \\ & + k \left[\gamma \frac{\partial (H'_y - v E'_z / \mu_0 c^2)}{\partial x} - \frac{\partial H'_x}{\partial y'} \right] \\ & = i \gamma J'_x + j J'_y + k J'_z + i \gamma \epsilon_0 \frac{\partial E'_x}{\partial t'} \\ & + j \epsilon_0 \gamma \frac{\partial (E'_y + v B'_z)}{\partial t} + k \epsilon_0 \gamma \frac{\partial (E'_z - v B'_y)}{\partial t}. \end{aligned} \quad (31)$$

Noting that $\epsilon_0 B = H/c^2$, noting that $1/\mu_0 c^2 = \epsilon_0$, and rearranging, we obtain

$$\begin{aligned} & i \left(\gamma \frac{\partial H'_z}{\partial y'} - \gamma \frac{\partial H'_y}{\partial z'} \right) \\ & + j \left[\frac{\partial H'_x}{\partial z'} - \gamma \left(\frac{\partial H'_z}{\partial x} + \frac{v}{c^2} \frac{\partial H'_z}{\partial t} \right) \right] \\ & + k \left[\gamma \left(\frac{\partial H'_y}{\partial x} + \frac{v}{c^2} \frac{\partial H'_y}{\partial t} \right) - \frac{\partial H'_x}{\partial y'} \right] \\ & = i \gamma J'_x + j J'_y + k J'_z + i \gamma \epsilon_0 \frac{\partial E'_x}{\partial t'} + j \gamma \epsilon_0 \\ & \cdot \left(\frac{\partial E'_y}{\partial t} + v \frac{\partial E'_y}{\partial x} \right) + k \gamma \epsilon_0 \left(\frac{\partial E'_z}{\partial t} + v \frac{\partial E'_z}{\partial x} \right), \end{aligned} \quad (32)$$

which, by (A16) and (A21), is

$$\begin{aligned} & i \gamma \left(\frac{\partial H'_z}{\partial y'} - \frac{\partial H'_y}{\partial z'} \right) + j \left(\frac{\partial H'_x}{\partial z'} - \frac{\partial H'_z}{\partial x} \right) \\ & + k \left(\frac{\partial H'_y}{\partial x} - \frac{\partial H'_x}{\partial y} \right) = i \gamma J'_x + j J'_y + k J'_z \\ & + i \gamma \epsilon_0 \frac{\partial E'_x}{\partial t'} + j \epsilon_0 \frac{\partial E'_y}{\partial t'} + k \epsilon_0 \frac{\partial E'_z}{\partial t'}. \end{aligned} \quad (33)$$

Comparing the x , y , and z components of the left side of (33) with those of the right side, we find that the

components have the same form as the components of (28) (the factor γ in the x components cancels if one equates only the individual components of the left and the right side of the equation). Thus the Cartesian components of Maxwell's equation (4) are invariant under relativistic transformations, but the vector equation itself is not invariant because γ is only present in the x components of (33) (therefore γ cannot be cancelled from (33), and (33) does not have the same form as (28) or (4)).

Note that in deriving (33) we used (A13) which only contains the electrostatic charge density.

3. Noninvariance of the First and the Fourth Original Maxwell's Equations under Relativistic Transformations Involving Lorentz-contracted Charges

As was shown in the preceding section, the invariance of Maxwell's equations (2) and (3) does not depend on the transformation equations for charge or current densities. Therefore in the discussion that follows we shall only consider Maxwell's equations (8) and (4).

By examining (12)–(15) we see that if we transform (8) by using (A24) (written explicitly for the Lorentz-contracted charge densities) instead of by using (A12), the transformed equation will not be of the form $\nabla' \cdot \mathbf{D}' = \rho'_0$. Therefore the original Maxwell's equation (8) (where ρ_0 is the electrostatic charge density) is not invariant under relativistic transformations involving Lorentz-contracted charges.

By examining (28)–(33) we see that the transition from (29) to (30) depends on (16) and therefore depends on the invariance of (8). Hence, if (8) is not invariant under a particular set of transformations, then (4) is also not invariant under the corresponding transformations. Therefore (4), just like (8), is not invariant under relativistic transformations involving Lorentz-contracted charges.

4. Invariance of the Relativistic Maxwell's Equations under Relativistic Transformations Involving Lorentz-contracted Charges

Starting with (9) and repeating the transformations used in (12)–(15), but employing now (A23) instead of

(A12), we find by inspection of (12)–(16) that (9) transforms into

$$\nabla' \cdot \mathbf{D}' = \rho'_0 (1 - u^2/c^2)^{-1/2}. \quad (34)$$

Thus (9), which is the relativistic form of the original Maxwell's equation (8) [or of (1)], is invariant under relativistic transformations involving Lorentz-contracted charge densities. Consequently, as explained in Sect. 3, (4) (in its scalar form; see Sect. 2) is also invariant under these transformations. And, of course, as explained in Sect. 2, (2) and (3) (in its scalar form; see Sect. 2) are also invariant. Hence the relativistic Maxwell's equations (9), (2), (3), and (4) (the last two in their scalar forms only) are all invariant under these transformations.

5. Discussion

As we have seen, contrary to the general perception, the invariance of Maxwell's equations under relativistic transformations is not clear-cut. First, which is a minor point, the equations $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$ and $\nabla \times \mathbf{H} = \mathbf{J} + \partial \mathbf{D}/\partial t$ are not invariant in their vector form. Second, which is very important, the original, prerelativistic, equation $\nabla \cdot \mathbf{D} = \rho$ (where $\rho = \rho_0$ is the electrostatic charge density) is only invariant if the charge density is transformed by means of (A12) (or by means of an equivalent equation). Third, the relativistic form $\nabla \cdot \mathbf{D} = \rho_0 (1 - u^2/c^2)^{-1/2}$ of the equation $\nabla \cdot \mathbf{D} = \rho$ is invariant provided that the charge density is transformed by means of (23) (or by means of an equivalent equation), where Lorentz contraction of moving charges is explicitly taken into account.

Therefore, if moving charges are Lorentz contracted, as is now generally believed, the prerelativistic Maxwell's equation $\nabla \cdot \mathbf{D} = \rho$, where ρ is the electrostatic charge density, is wrong because it does not satisfy the requirement that physical laws must have the same form in all inertial reference frames (Einstein's principle of relativity). The relativistically correct Maxwell's equations must then be

$$\nabla \cdot \mathbf{D} = \rho_0 (1 - u^2/c^2)^{-1/2}, \quad (35)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (36)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (37)$$

and

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}. \quad (38)$$

Clearly, if accepted^b, the relativistic modification of the equation $\nabla \cdot \mathbf{D} = \varrho$ into $\nabla \cdot \mathbf{D} = \varrho_0 (1 - u^2/c^2)^{-1/2}$ will have profound effects on the theory of many electromagnetic phenomena involving rapidly moving electric charges. What is especially important, this modification will affect not only the theory of charges moving at constant velocities (the domain of the special theory of relativity), but also the theory of charges moving with acceleration.

We can obtain an insight into some of the consequences of replacing $\nabla \cdot \mathbf{D} = \varrho$ by $\nabla \cdot \mathbf{D} = \varrho_0 (1 - u^2/c^2)^{-1/2}$ by examining the following equations representing the general solution of Maxwell's equations for a vacuum [3]

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int \left\{ \frac{[\varrho]}{r^3} + \frac{1}{r^2 c} \left[\frac{\partial \varrho}{\partial t} \right] \right\} \mathbf{r} dV' - \frac{1}{4\pi\epsilon_0 c^2} \int \frac{1}{r} \left[\frac{\partial \mathbf{J}}{\partial t} \right] dV', \quad (39)$$

$$\mathbf{H} = \frac{1}{4\pi} \int \left\{ \frac{[\mathbf{J}]}{r^3} + \frac{1}{r^2 c} \left[\frac{\partial \mathbf{J}}{\partial t} \right] \right\} \times \mathbf{r} dV'. \quad (40)$$

These equations have been obtained by assuming that the charge density ϱ in (1) is the electrostatic charge density (does not depend on the velocity of the charge) and by assuming that $\mathbf{J} = \varrho \mathbf{u}$ ^c. Therefore, in terms of the notations used in this paper, ϱ and \mathbf{J} in these equations are $\varrho = \varrho_0$ and $\mathbf{J} = \varrho_0 \mathbf{u}$, respectively, and the equations can be written as

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int \left\{ \frac{[\varrho_0]}{r^3} + \frac{1}{r^2 c} \left[\frac{\partial \varrho_0}{\partial t} \right] \right\} \mathbf{r} dV' - \frac{1}{4\pi\epsilon_0 c^2} \int \frac{1}{r} \left[\frac{\partial (\varrho_0 \mathbf{u})}{\partial t} \right] dV', \quad (41)$$

^b I am not aware of any publications expressing Maxwell's equation $\nabla \cdot \mathbf{D} = \varrho$ (which represents Gauss's law in its differential form) as $\nabla \cdot \mathbf{D} = \varrho_0 (1 - u^2/c^2)^{-1/2}$. It is remarkable that although most authors of books on electromagnetic theory and relativity emphasize the idea of Lorentz contraction, they express Gauss's law as $\nabla \cdot \mathbf{D} = \varrho$, with a clear implication that ϱ is the electrostatic charge density, and do not even mention the possibility of expressing Gauss's law as $\nabla \cdot \mathbf{D} = \varrho_0 (1 - u^2/c^2)^{-1/2}$, where the Lorentz contraction of moving charges is taken into account.

^c Equations (39) and (40) are closely related to the retarded potential equations

$$\varphi = \frac{1}{4\pi\epsilon_0} \int \frac{[\varrho]}{r} dV' \quad (*39)$$

and

$$\mathbf{A} = \frac{\mu_0}{4\pi} \int \frac{[\mathbf{J}]}{r} dV', \quad (*40)$$

which were first obtained (in a different notation) by Ludwig Lorenz in 1867, and the meaning of ϱ and \mathbf{J} in (39) and (40) is the same as that in Lorenz's equations (*39) and (*40).

and

$$\mathbf{H} = \frac{1}{4\pi} \int \left\{ \frac{[\varrho_0 \mathbf{u}]}{r^3} + \frac{1}{r^2 c} \left[\frac{\partial (\varrho_0 \mathbf{u})}{\partial t} \right] \right\} \times \mathbf{r} dV'. \quad (42)$$

However, if ϱ in Maxwell's equation $\nabla \cdot \mathbf{D} = \varrho$ is the Lorentz-contracted charge density $\varrho = \varrho_0 (1 - u^2/c^2)^{-1/2}$, (39) and (40) become

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \int \left\{ \frac{[\varrho_0 (1 - u^2/c^2)^{-1/2}]}{r^3} + \frac{1}{r^2 c} \left[\frac{\partial \{\varrho_0 (1 - u^2/c^2)^{-1/2}\}}{\partial t} \right] \right\} \mathbf{r} dV' - \frac{1}{4\pi\epsilon_0 c^2} \int \frac{1}{r} \left[\frac{\partial \{\varrho_0 \mathbf{u} (1 - u^2/c^2)^{-1/2}\}}{\partial t} \right] dV'. \quad (43)$$

and

$$\mathbf{H} = \frac{1}{4\pi} \int \left\{ \frac{[\varrho_0 \mathbf{u} (1 - u^2/c^2)^{-1/2}]}{r^3} + \frac{1}{r^2 c} \left[\frac{\partial \{\varrho_0 \mathbf{u} (1 - u^2/c^2)^{-1/2}\}}{\partial t} \right] \right\} \times \mathbf{r} dV'. \quad (44)$$

The difference between (41), (42) and (43), (44) is profound. The electric and magnetic fields given by (41) and (42) are quite distinct from the fields given by (43) and (44). In particular, the presence of additional u 's in (43) and (44) makes the time derivatives in (43) and (44) completely different from those in (41) and (42). This means, among other things, that the electric and magnetic fields for charges in accelerated motion (and therefore all radiation fields and associated effects) computed from (43) and (44) are fundamentally different from the fields computed from (41) and (42).

It is important to emphasize that (39) and (40) are the general solutions of Maxwell's equations. Therefore the differences manifested by comparing (41) and (42) with (43) and (44) will be inevitably replicated in all formulas and equations for rapidly moving charges (regardless of the actual method of derivation) depending on whether these formulas and equations are based on the pre-relativistic Maxwell's equations or on Maxwell's equations corrected for Lorentz contractions of moving charges.

But can we be absolutely certain that the relativistic modification of the equation $\nabla \cdot \mathbf{D} = \varrho$ and the corresponding modification of the various electrodynamic equations are necessary? It has been recently shown [4] that the main equations of the special theory of relativity can be obtained without invoking Lorentz contraction of moving bodies. Similarly, the relativistically correct

expressions for the electric and magnetic fields of moving point and line charges can be obtained without assuming that moving charges are Lorentz-contracted [5, 6]. Furthermore, it is known that the relativistic transformation equations for all electromagnetic quantities can be derived from the prerelativistic Maxwell's equations by assuming that these equations have the same form in all inertial frames [7]. It is also well known that the physical significance of Lorentz contraction has been the subject of considerable controversy and re-interpretation [8]. However, the controversy and the re-interpretation have not at all affected the theory of relativity as such. Therefore, although the idea of Lorentz contraction is important in Einstein's approach to the formulation of the special relativity theory, it does not appear to be an indispensable element of the theory of relativity itself.

There may be a simple explanation why the controversy about Lorentz contraction and the different re-interpretations of its physical significance have had no effect on the theory of relativity. If one accepts the theory of relativity as the body of equations, methods, and techniques whereby physical quantities measured in one inertial reference frame can be correlated with physical quantities measured in any other inertial reference frame, then the interpretation of this or that relativistic formula or equation does not affect the theory, since a mere interpretation of a formula or equations does not change the formula or equations. Therefore the fact that the original Maxwell's equations are not invariant under relativistic transformations involving Lorentz-contracted charge densities may not necessarily mean that the original Maxwell's equations are incorrect, especially in view of the fact that they are invariant under relativistic transformations not involving Lorentz-contracted charge densities. This brings us to (A11) and (A12). If interpreted as relations between proper charge densities, these equations are wrong because proper charge densities are invariant by definition. However, if interpreted as charge density transformation equations disregarding Lorentz contraction (thereby eliminating the distinction between proper and nonproper charge densities), they cannot be objectively rejected unless the reality of Lorentz contraction is proved beyond any doubt^d.

Thus the above analysis of the relativistic invariance of Maxwell's equation presents us with a dilemma: ei-

ther we accept Maxwell's equations in their original form and once again question the physical significance of Lorentz contraction, or we modify Maxwell's equations and question the validity of many formulas and equations derived from the original Maxwell's equations. At this time it is hardly possible to resolve the dilemma on the basis of theoretical considerations. In all probability the resolution of the dilemma will come from new experiments with rapidly moving charges and from comparing the two alternative theoretical models with the experimental data.

Acknowledgements

I am pleased to express my gratitude to Professor A. M. Portis of University of California for communicating to me (9) and (A22), and for other valuable information

Appendix [9]

$$E_x = E'_x, \quad (\text{A } 1)$$

$$E_y = \gamma (E'_y + v B'_z), \quad (\text{A } 2)$$

$$E_y = E'_y / \gamma + v B_z, \quad (\text{A } 3)$$

$$E_z = \gamma (E'_z - v B'_y), \quad (\text{A } 4)$$

$$E_z = E'_z / \gamma - v B_y, \quad (\text{A } 5)$$

$$B_x = B'_x, \quad (\text{A } 6)$$

$$B_y = \gamma (B'_y - v E'_z / c^2), \quad (\text{A } 7)$$

$$B_y = B'_y / \gamma - v E_z / c^2, \quad (\text{A } 8)$$

$$B_z = \gamma (B'_z + v E'_y / c^2), \quad (\text{A } 9)$$

$$B_z = B'_z / \gamma + v E_y / c^2, \quad (\text{A } 10)$$

$$\varrho_0 = \gamma [\varrho'_0 + (v/c^2) J'_x], \quad (\text{A } 11)$$

$$\varrho_0 = \varrho'_0 / \gamma + (v/c^2) J_x, \quad (\text{A } 12)$$

$$J_x = \gamma (J'_x + v \varrho'_0), \quad (\text{A } 13)$$

$$J_y = J'_y, \quad (\text{A } 14)$$

$$J_z = J'_z, \quad (\text{A } 15)$$

$$\frac{\partial}{\partial x'} = \gamma \left(\frac{\partial}{\partial x} + \frac{v}{c^2} \frac{\partial}{\partial t} \right), \quad (\text{A } 16)$$

$$\frac{\partial}{\partial x} = \frac{1}{\gamma} \frac{\partial}{\partial x'} - \frac{v}{c^2} \frac{\partial}{\partial t}, \quad (\text{A } 17)$$

$$\frac{\partial}{\partial y} = \frac{\partial}{\partial y'}, \quad (\text{A } 18)$$

^d It is important to emphasize that the modification of Maxwell's equation $\nabla \cdot \mathbf{D} = \varrho$ into $\nabla \cdot \mathbf{D} = \varrho_0 (1 - u^2/c^2)^{-1/2}$ does not depend on (A11) and (A12) or on their interpretation. Therefore, the considerations concerning (39)–(44) presented above remain valid regardless of the meaning or validity of (A11) and (A12).

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z'},$$

(A 19)

$$\frac{\partial}{\partial t} = \gamma \left(\frac{\partial}{\partial t'} - v \frac{\partial}{\partial x'} \right),$$

(A 20)

$$\frac{\partial}{\partial t'} = \gamma \left(\frac{\partial}{\partial t} + v \frac{\partial}{\partial x} \right).$$

(A 21)

The following equations are obtained from (A 11)–(A 13) by writing them explicitly for Lorentz-contracted charge densities

$$\varrho_0 (1 - u^2/c^2)^{-1/2} = \gamma [\varrho'_0 (1 - u'^2/c'^2)^{-1/2} + (v/c^2) J'_x], \quad (\text{A } 22)$$

$$\varrho_0 (1 - u^2/c^2)^{-1/2} = \varrho'_0 (1 - u'^2/c'^2)^{-1/2} / \gamma + (v/c^2) J_x, \quad (\text{A } 23)$$

$$J_x = \gamma [J'_x + v \varrho'_0 (1 - u'^2/c'^2)^{-1/2}]. \quad (\text{A } 24)$$

- [1] See, for example, W. G. V. Rosser, *Interpretation of Classical Electromagnetism*; Kluwer, Dordrecht 1997, pp. 1–54.
- [2] See, for example, A. M. Portis, *Electromagnetic Fields: Sources and Media*; Wiley & Sons, New York 1978, pp. 383–384.
- [3] See, for example, J. D. Jackson, *Classical Electrodynamics*, 3rd ed., Wiley & Sons, New York 1999, pp. 246–247; W. G. V. Rosser, *Interpretation of Classical Electromagnetism*; Kluwer, Dordrecht 1997, pp. 48–49, 82–84; M. A. Heald and J. B. Marion, *Classical Electromagnetic Radiation*; Saunders, Fort Worth 1995, pp. 261–262; O. D. Jefimenko, *Electricity and Magnetism*, 2nd ed.; Electret Scientific, Star City 1992, pp. 514–516.
- [4] Oleg D. Jefimenko, Retardation and relativity: Derivation of Lorentz-Einstein transformations from retarded integrals for electric and magnetic fields, *Amer. J. Phys.* **63**, 267 (1995).
- [5] Oleg D. Jefimenko, Direct calculation of electric and magnetic fields of an electric point charge moving with constant velocity, *Amer. J. Phys.* **62**, 79 (1994).
- [6] Oleg D. Jefimenko, Retardation and relativity: The case of a moving line charge, *Amer. J. Phys.* **63**, 454 (1995).
- [7] See W. G. V. Rosser, *Interpretation of Classical Electromagnetism*; Kluwer, Dordrecht 1997, pp. 375–383.
- [8] See, for example, J. Terrell, Invisibility of the Lorentz Contraction, *Phys. Rev.* **116**, 1014 (1959); Roy Weinstein, Observation of length by a single observer, *Amer. J. Phys.* **28**, 607 (1960); V. F. Weisskopf, The visual appearance of rapidly moving objects, *Phys. Today* **13**, 24 (1960); A. Gamba, Physical quantities in different reference systems according to relativity, *Amer. J. Phys.* **35**, 83 (1967); G. D. Scott and M. R. Viner, The geometrical appearance of large objects moving at relativistic speeds, *Amer. J. Phys.* **33**, 534 (1965); V. N. Strel'tsov, On the relativistic length, *Found. Phys.* **6**, 293 (1976); Kevin G. Suffern, The apparent shape of a moving sphere, *Amer. J. Phys.* **56**, 729 (1988); V. N. Strel'tsov, The Question is: Are Fast-Moving Scales Contracted or Elongated?, *Hadronic J.* **17**, 105 (1994).
- [9] See, for example, Oleg D. Jefimenko, *Electromagnetic Retardation and Theory of Relativity*; Electret Scientific, Star City 1997, pp. 150–158.